Title: "Path integrals, finite temperatures and lattices"

Abstract: "Surprisingly, partition functions for some model systems in statistical mechanics are invariant under formally reflecting the sign of temperature, $T: +T \to -T$. We call this T-reflection invariance. Clearly, partition functions for generic statistical systems cannot be invariant under T-reflection. However, in this talk we focus on finite-temperature path integrals and give a general picture for why finite-temperature path integrals in quantum field theory should ~~be invariant~~ behave well under T-reflection. We probe this general picture in the context of harmonic oscillators (in one-dimension) and in [chiral] conformal field theories (in two-dimensions) and in the mathematics of modular forms. We find that the relevant path integrals are often ~~not~~ invariant only up to overall T-independent phases, which could be naturally interpreted as a new anomaly under large coordinate transforms [redundant ones]."

$\boxed{\text{Kadanoff Seminar @ U. Chicago, 25 Sept. 2018 [Tuesday]}}$

$\boxed{\text{David A. McGady}}$ $\boxed{\text{Based off of 1711.07536, 1806.0987\{3/4/5\}}}$

$\boxed{\text{McGill University, Center for High Energy Physics, 03 Oct. 2018 [Wednesday]}}$

I. Introduction and Outline: $\boxed{\text{I.1}}$

$\boxed{\text{A.}}$ The story (for today): simple and straightforward, though not complete and not unimpeachable.

⊛ Many PF's satisfy $Z_{PF}(\beta) = e^{i\gamma_R} Z_{PF}(-\beta)$. Very surprising.

→ IF we fix E_{VAC}!

→ T reflection ⟺ E_{VAC} fixed!

→ But why? Where should this come from? Why covariance?

(Why anything?)

⊛ My answer: IF $Z_{PF}(\beta) = Z_{QFT}(\beta)$, THEN $+\beta \leftrightarrow -\beta$ should describe the same finite-T QFT path integral.

→ Redundant descriptions of finite-T/finite-β.

→ General argument! (Worryingly so?)

→ Very close cousin to argument for modularity ∈ 2d CFT!

⊛ HOWEVER!!!... However, $Z_{QFT}(\beta) \to e^{i\gamma_R} Z_{QFT}(\beta)$ as $\beta \to -\beta$ w/ $e^{i\gamma_R} \neq 1$ and $|e^{i\gamma_R}| = 1$ is a dependence on redundant sign!

→ $e^{i\gamma_R} \neq 1$ ⟺ Resembles unphysical redundancy ∈ Path Int.

→ $e^{i\gamma_R} \neq 1$ ⟺ Resembles an anomaly phase.

→ Will show $e^{i\gamma_R}$ ⟷ Fujikawa ⟷ (KK) zero-modes ⟷ mod wt.

$\boxed{B.}$ Examples: restrict our attention to purely discrete (elsewhere, also, purely continuum) examples.

- ⊛ Finite-spectrum w/ no symmetry ⇒ no T-reflection inv.!
- ⊛ Finite-spectrum w/ symmetry ⇒ E_{VAC}-tuned ⇒ T-reflection inv.!
- ⊛ Harmonic oscillator: bounded below, but not above. Yet...

$\boxed{C.}$ Harmonic oscillator: $E_n^{(\Delta)} = \Delta + \omega(n+\frac{1}{2}) \Rightarrow Z^{(\Delta)}(\beta) = \dfrac{e^{-\beta\Delta}}{2\sinh(\beta\omega/2)}$

- ⊛ $\Delta := 0 \Rightarrow Z_{HO}(\beta) = \dfrac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}} = \dfrac{1}{2\sinh(\beta\omega/2)} = -Z_{HO}(-\beta)$!
- ⊛ Oddness ⇔ zero-mode along the S^1_β assoc. finite-β!
- ⊛ Many oscillators: $Z(\beta) = \prod_n Z_{HO}(\omega_n|\beta)^{d_n} \to (-1)^{\sum_n d_n} Z(\beta)$
- ⊛ [Free (2nd)] QFT as "tons of oscillators" ⇒ check E_{VAC}... WORKS!!!

$\boxed{D.}$ General argument: $\boxed{\text{⊛ Finite-}\beta \in \text{QFT} \Rightarrow t_E \sim t_E + n\beta. \Leftrightarrow S^1_\beta \text{ !!}}$

- ⊛ $Z_{QFT}(\beta) = \int D\phi \; e^{-\int_{M_{d-1}\times S^1_\beta} d^dx \, \mathcal{L}_E[\phi(x)]}$
- ⊛ $Z_{QFT}(\beta)$ is/can only be function of lattice of points $\Lambda(\beta) = \{m\beta \,|\, m \in \mathbb{Z}\}$.
- ⊛ $Z_{QFT}(\beta) \to e^{i\gamma_R} Z_{QFT}(\beta)$ as $\beta \to -\beta$
- ⊛ Again, what is $e^{i\gamma_R}$?

$\boxed{E.}$ Main results: For (0+1)d QFTs ~~w/~~ & (1+1)d χCFTs, mainly:

⊛ $e^{i\gamma_R} = (-1)^{R.\Sigma}$; KK-zero-modes, assoc. w/ S^1_β. $\boxed{\text{General}}$

⊛ $\cancel{\chi}$ 2d CFTs: $e^{i\gamma_R} = (-1)^{R.\Sigma} = (-1)^k$, w/ k = modular weight!

⊛ $(\beta|\tau) \leftrightarrow (-\beta|-\tau)$ simply a redundant way to describe the same compact geometry. Non-invariance $\Leftarrow \boxed{e^{i\gamma_R} \neq 1}$...

⊛ T-Reflection seems robust; seems to fix E_{VAC} sometimes!

$\boxed{F.}$ Outline: ⊛ II. HO: N-particle Fock-space, Path Integrals, KK modes, the lattice and continuations/measure variations $\boxed{1711.\alpha\beta\gamma\delta\varepsilon}$

⊛ III. Lattices and Σ-rules: $E_k(z)$ & $S+T$, $E_k(z)$ & R, Ring of MFs, $\boxed{1806.\{\alpha\beta\gamma\delta\}3}$ + $\boxed{1806.\{\alpha\beta\gamma\delta\}4}$ $\prod(1-q^n)^{d(n)} \leftrightarrow \sum n^{-s} d(n)$, and Mellin/L-functions.

⊛ IV. 2d χCFTs and lattices: ~~Forms~~ $\Leftrightarrow$ Lattice, scale inv. $\Rightarrow$ modularity/, $\boxed{\text{Chicago}}$ $\boxed{1806.\{\alpha\beta\gamma\delta\}3}$ + $\boxed{1806.\{\alpha\beta\gamma\delta\}5}$ Redundancies, $\boxed{\frac{1}{\eta(\tau)} \text{ and phases}}$, and GGAs $\boxed{a(\theta) \text{ \& } L(\theta) \text{ and automorphisms}}$ $\leftarrow \boxed{\text{McGill}}$

⊛ V. ...and everything else! Continuations? SPT phases? Prob. Thy? ~~Hopf~~? Homomorphisms? H^*? Vacuum energies? Much promise! Much to do!

$\boxed{G.}$ Acknowledgements: ⊛ Thanks for having me!

⊛ JFRD, S~~M~~, SR, and MY!

⊛ AC! Others!

⊛ JSPS @ IPMU & Carlsberg @ NBIA<NBI!

II. Harmonic Oscillator: $\boxed{\text{II}.1}$

$\boxed{\text{A.}}$ Consider a non-interacting QFT of a free scalar w/ rest energy ω on the (0+1)d manifold S^1_β.

- ⊛ The N-particle Fock space consists of ONE state: $|\omega^{(1)}, \dots, \omega^{(N)}\rangle$
- ⊛ Thus $Z^{(N)}_{CAN}(\beta) = (e^{-\beta\omega})^N \Rightarrow$ Not T-reflection invariant
- ⊛ However, $Z_{GC}(\beta) = e^{-\beta E_{VAC}} \sum_{N=0}^{\infty} Z^{(N)}_{CAN}(\beta) = \frac{e^{-\beta E_{VAC}}}{1-e^{-\beta\omega}}$, can be!

[redacted] $\boxed{\text{If } E_{VAC} = \frac{\omega}{2}\text{, we have } e^{i\gamma_A} = -1\text{: VIA } q \to \frac{1}{q} \text{ in } \frac{q^{1/2}}{1-q}.}$

$\boxed{\text{B.}}$ Similarly, we can compute the path integral for this QFT.

- ⊛ $Z(\beta) = \int D\phi \, e^{-\int_0^\beta dt_E \mathcal{L}_E[\phi(x)]}$, w/ $\mathcal{L}_E = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}\omega^2\phi^2$
- ⊛ $= \dots = \frac{1}{\omega\beta} \prod_{n \geq 1} \frac{1}{(\omega\beta + in\pi)} \frac{1}{(\omega\beta - in\pi)} = \prod_{n \in \mathbb{Z}} \frac{1}{\omega\beta + in\pi}$ ← decoupled KK modes (free)
- ⊛ $Z(\beta) \to -Z(\beta)$ as $\beta \to -\beta$, formally, from unpaired nature of KK <u>zero-mode</u>, $\frac{1}{\beta}$.

$\boxed{\text{C.}}$ <u>General point:</u> ⊛ fixed particle processes/sectors $\neq$ full QFT.

- ⊛ Only PFs w/ access to Full Fock Space can match the QFT path integral
- ⊛ QFT path integral "should" be "OK" under T-reflection...
- ⊛ ... unlike Stat Mech PFs.
- ⊛ Here: IF $E_{VAC} = \frac{\omega}{2}$ (normal ordering), THEN $Z_{GC}(\beta) = Z(\beta)$

$\boxed{\text{D.}}$ The lattice, again: $t_E \sim t_E + n\beta \Rightarrow Z(\beta) = Z(\Lambda(\beta))$ w/

$\Lambda(\beta) = \{m\beta \mid m \in \mathbb{Z}\}$ for QFT path integrals.

* $\Lambda(-\beta) = \Lambda(\beta)$, and ~~and in KK picture~~ we expect $Z(\beta) = Z(-\beta)$
* ... and we see that for the QFT path integral $Z(\beta) = -Z(-\beta)$
* ... follows very easily from looking at KK momentum decomp...
* ... up to this $(-1) = e^{i\gamma_R}$ coming from the KK zero-mode!
* Again: $\beta \Leftrightarrow -\beta$ a redundancy in BC $t_E \sim t_E + n\beta$ $\forall n$ [illegible]
  and $e^{i\gamma_R} \neq 1$ resembles an anomaly $\Leftrightarrow$ KK zero-mode!

$\boxed{\text{E.}}$ Variation of the Path Integral Measure and $e^{i\gamma_R}$:

* Focus on $\frac{1}{\beta} \Leftrightarrow$ KK zero-mode, and how it evolves as $\beta \to e^{i\theta}\beta \to -\beta$ continuously.
* $Z_0(\beta, \omega) = \int_{-\infty}^{\infty} dy_0\, e^{-S_E[y_0, \beta]}$, w/ y_0 the Fourier coefficient for the constant mode $\phi(t_E) = \text{const}$ on $[0, \beta]$.
* $$= \dots = \int_{-\infty}^{\infty} dy_0\, e^{-(\beta\omega y_0)^2} \propto \sqrt{\frac{1}{\beta^2\omega^2}}$$
* Use path of steepest ascent: $\beta \to e^{i\theta}\beta$ requires us to choose $\int_{\Gamma_\theta}$ in (now) complexified field-space.
* Choose Γ_θ $\ni$: $e^{-(\beta e^{i\theta} y_0(\theta))^2} \to 0$ maximally fast. Easy to see $\Gamma_\theta = e^{-i\theta}\mathbb{R}$
* $$\Rightarrow Z_0(\beta e^{i\theta}, \omega) = \int_{-\infty}^{\infty} \boxed{d(e^{-i\theta} y_0)}\, e^{-(\beta e^{i\theta})^2 (\omega^2)(y_0 e^{-i\theta})^2} = e^{-i\theta} Z_0(\beta, \omega) \to -Z_0(\beta, \omega)!$$

$\boxed{\text{F.}}$ Thm: $Z_0(\beta, w) \to Z_0(e^{i\theta}\beta, w) = e^{-i\theta} Z_0(\beta, w) \to -Z_0(\beta, w_0)$

as $\beta \to e^{i\theta}\beta \qquad\qquad \to -\beta$

via $\int_{-\infty}^{\infty} d\gamma_0 \longrightarrow \int_{\Gamma_\theta} d\gamma_0(\theta) = \int_{-\infty}^{\infty} d(e^{-i\theta}\gamma_0) = e^{-i\theta}\int_{-\infty}^{\infty} d\gamma_0 \to -\int_{-\infty}^{\infty} d\gamma_0$

i.e. via a variation in the path integral measure.

$\boxed{\text{G.}}$ Summary: ⊛ $Z^{(N)}(\beta)$ - not T-reflection inv.

⊛ $Z_{GC}(\beta)$ MAY be, as $Z_{GC}(\beta) \sim Z(\beta)$

⊛ $Z(\beta) \to e^{i\gamma_R} Z(\beta)$ as $\beta \to -\beta$ should have

$e^{i\gamma_R} = 1$ as $\beta \leftrightarrow -\beta$ describe the SAME periodicity!

⊛ $Z(\beta) \to e^{i\gamma_R} Z(\beta)$ as $\beta \to -\beta$, here, has

$e^{i\gamma_R} = (-1)^{\sum \text{KK-zero-modes}}$.

⊛ AND $e^{i\gamma_R}$ = variation in measure of path integral as $\beta \to -\beta$

⇓

- Very general story.
- Would like to probe somewhere where ~~the~~ things are "clear"

III. Lattices and Sums-Rules : $\boxed{\text{III.1}}$

$\boxed{\text{A.}}$ **Note**: even free QFTs in $d>1$-dim will be MUCH more complicated, if we want to send $\beta \to -\beta$ "smoothly".

⊛ Free massless scalar on $M_2 = S^1_L \times S^1_\beta$. Momenta $p_n = \frac{2\pi n}{L}$; $E_n = p_n$.

⊛ $Z^{(1)}_{CAN}(\beta) = \sum_{n=0}^{\infty} e^{-\beta(2\pi n/L)} = \frac{1}{1-e^{-2\pi\beta/L}} := \frac{1}{1-q}$

⊛ $Z^{(M)}_{CAN}(\beta) = \prod_{n=1}^{M} \frac{1}{1-q^n}$

⊛ $\boxed{Z_{GC}(\beta) = \left(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right) q^{E_{VAC}}}$ → not obvious how to continue on first approach.

⊛ But $Z(\beta) = \int D\phi\, e^{-S_E[\phi]}$ looks better!

$\boxed{\text{B.}}$ For this case, we already expect $Z(-\beta) \to e^{i\gamma_R} Z(\beta)$ as $\beta \to -\beta$, from an appeal to lattice of identical points $\Lambda(-\beta) = \Lambda(\beta)$.

$\boxed{\text{C.}}$ Here, we have a 2d lattice, w/ vectors "L" and "β" that are not collinear; a non-degenerate lattice on the plane.

$\boxed{\text{D.}}$ Path integral, by definition, is a function of this lattice — at least this is what we would expect.

⇓

Here, we will discuss some other functions of 2d lattices / lattices on the complex plane!

$\boxed{\text{E.}}$ Consider $\boxed{\boxed{\tau \in \mathbb{C}} \text{ w/ } \boxed{\operatorname{Im}(\tau) \neq 1}}$, and consider the toroidal compactification $\boxed{z \sim z + m + n\tau}$ for $z \in \mathbb{C}$. This defines a lattice $\boxed{\Lambda(\tau) := \{ m + n\tau \mid m, n \in \mathbb{Z} \}}$.

$\boxed{\text{F.}}$ Now, consider $E_k(\tau) := \dfrac{1}{2\zeta(k)} \displaystyle\sum'_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^k}$. Clearly, this is a function of $\Lambda(\tau)$. Now, note:

- ⊛ $\Lambda(\tau) = \tau\Lambda(-1/\tau) = \Lambda(\tau+1) = \Lambda(-\tau)$
- ⊛ $E_k(\tau) = \tau^k E_k(-1/\tau) = E_k(\tau+1) = E_k(-\tau)$
- ⊛ These have names: $S: \tau \to -\frac{1}{\tau}$, $T: \tau \to \tau+1$, and $R: \tau \to -\tau$.

  Together, S & T generate $SL_2(\mathbb{Z})$; S, T, and R generate $GL_2(\mathbb{Z})$.

$\boxed{\text{G.}}$ It's a straightforward bit of algebra (involving $\displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{z+n} = \frac{\pi}{\tan \pi z}$ and $\frac{d}{dz}$'s) to show

$$E_k(\tau) = E_k(\tau+1) \Rightarrow E_k(\tau) = 1 + \sum_{n=1}^{\infty} \frac{2\sigma_{k-1}(n)}{\zeta(1-k)} \left(e^{2\pi i \tau}\right)^n$$

- ⊛ Analogous to $Z(\tau) = q^{-\Delta}\left(\displaystyle\sum_{n=0}^{\infty} c(n) q^n\right)$, $q := e^{2\pi i\tau} = e^{-2\pi\beta/L}$
- ⊛ Because $E_k(\tau) = 1 + \mathcal{O}(q)$ we can formally write

$$E_k(\tau) = \prod_{n=1}^{\infty} \left(1-q^n\right)^{d_k(n)} \quad \text{for } d_k(n).$$

- ⊛ [Note: for k = odd $E_k(\tau) = 0$; for $k > 6$, $c_k(n)$ and $d_k(n)$'s are rational but not generally integer. But ~~[illegible]~~ more gen. sly.]

$\boxed{\text{H.}}$ Now, for $\tau \to -\tau$ and FORMALLY $q \to \frac{1}{q}$:

⊛ Just as we could see $Z(\beta) \to -Z(\beta)$ as $\beta \to -\beta$ for HO by looking @ $q \to \frac{1}{q}$ in $Z(\beta) = \frac{q^{1/2}}{1-q}$, we can try to formally do the same here.

⊛ $E_k(-\tau) = E_k(+\tau)$ and $E_k(\tau) = \prod_{n=1}^{\infty} (1-q^n)^{d_k(n)}$ suggests two sum-rules: ⊛ $(1-q^n)^{d_k(n)} \to (1-q^{-n})^{d_k(n)} = (q^{-n})^{d_k(n)} (-1)^{d_k(n)} ((1-q^n)^{d_k(n)})$

$$\left.\begin{aligned} &\longrightarrow \Rightarrow R.\sum_{n=1}^{\infty} (-n)^1 \cdot d_k(n) = 0 \\ &\longrightarrow \Rightarrow R.\sum_{n=1}^{\infty} (-n)^0 \cdot d_k(n) = k \end{aligned}\right\} \Rightarrow \text{Special value of some zeta-function,}$$

⊛ What is this zeta-function? How to "test" this "prediction" of T-reflection? The point/motivation for $\boxed{1806.(\alpha\beta\gamma\delta)4}$

$\boxed{\text{I.}}$ Note two things! (Or three...)

⊛ $$\int_0^{\infty} \frac{dt}{t} t^s \sum_{n=1}^{\infty} c(n)\, e^{-2\pi n t} = \int_0^{\infty} \frac{dt}{t} t^s f(it) = \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

⊛ $$-\frac{1}{2\pi i}\frac{d}{d\tau} \log E_k(\tau) = \sum_{n=1}^{\infty} \left(\sum_{m|n} m\, d_k(m)\right) q^n$$

⊛ Together, we see that the "zeta functions" that controls these two sum-rules is ALMOST $\frac{(2\pi)^s}{\Gamma(s)} \cdot \int_0^{\infty} \frac{dt}{t} t^s \left(\frac{d}{dt} \log E_k(it)\right)$!

$\boxed{J}$ Call $E_k(\tau) := Z(\tau)$, and call $\frac{1}{2\pi i}\partial_\tau \log Z(\tau) := \langle T(\tau)\rangle$.

Further, call $L^*_{\langle T\rangle}(s) := \int_0^\infty \frac{dt}{t} t^s \langle T(it)\rangle$.

⊛ Now, $E_k(-\frac{1}{\tau}) = \tau^k E_k(\tau) \Rightarrow$ same for $Z(\tau)$

⊛ Thus, $\langle T(-\frac{1}{\tau})\rangle = \tau^2 \langle T(\tau)\rangle$ + an anomaly piece coming from

the fact that $\frac{\partial_\tau \, \tau^k E_k(\tau)}{\tau^k E_k(\tau)} = \frac{\partial_\tau E_k(\tau)}{E_k(\tau)} + \frac{k}{\tau}$

$\boxed{K}$ Using $\tau = it$, and setting $-\frac{1}{it} = \frac{i}{t}$, we can break-up

$\int_{0=t}^{\infty=t} = \int_{0=t}^{1=t} + \int_{1=t}^{\infty=t}$ and then use $\tau \to -\frac{1}{\tau}$ to relate

the integrands...

$\boxed{L}$ In the end, we find

$$L^*_{\langle T\rangle}(s) = \frac{k}{2\pi}\frac{1}{s-1} + \text{entire}(s)$$

⊛ $\frac{1}{s-1} \leftrightarrow \int_1^\infty \frac{dt}{t} t^{s} \left(\frac{k}{it}\right)$, the "anomaly" term in $\langle T(-\frac{1}{\tau})\rangle$

⊛ "entire(s)" comes from the fact that $\int_1^\infty \frac{dt}{t} t^s \sum_n c(n) e^{-2\pi n t}$

is a sum of $\sum_n \frac{c(n)}{(2\pi n)^s} \int_{t=2\pi n}^{t=\infty} \frac{dt}{t} t^s e^{-t}$, which decays

exponentially quickly.

$\boxed{\text{M.}}$ Thus, we see from the formal manipulation

$$L^*_{\langle T\rangle}(s) = \int_0^\infty \frac{dt}{t}\, t^s \left(\sum_{n=1}^\infty \left(\sum_{m|n} m\, d_k(m)\right) e^{-2\pi n t}\right)$$

$$\Big\downarrow \sim \frac{\Gamma(s)}{(2\pi)^s}\, R.\sum_{n=1}^\infty \left(\sum_{m|n} m\, d_k(m)\right) n^{-s}$$

$\boxed{\text{N}}$ So we <u>almost</u> got the sum rule if we write/define

$$R.\sum_{n=1}^\infty \left(\sum_{m|n} m\, d_k(m)\right) n^{-s} := \frac{(2\pi)^s}{\Gamma(s)} L^*_{\langle T\rangle}(s),$$

and then note

(*) $\left(\sum_{m|n} m\, d_k(m)\right) n^{-s} \sim n^{-(s-1)} d_k(n)$, and write

$$R.\sum_{n=1}^\infty d_k(n)\, n^{-s} := \frac{1}{\zeta(s+1)}\, R.\sum_{n=1}^\infty \left(\sum_{m|n} m\, d_k(m)\right) n^{-(s+1)}$$

(*) Then note

- $$\lim_{s\to -1} R.\sum_{n=1}^\infty d_k(n)\cdot n^{-s} = \lim_{s\to -1} \frac{1}{\zeta(s+1)} \frac{(2\pi)^{s+1}}{\Gamma(s+1)}\left(\frac{k}{s} + \text{entire}\right) = 0 \quad ✓$$

- $$\lim_{s\to 0} R.\sum_{n=1}^\infty d_k(n)\cdot n^{-s} = \lim_{s\to 0} \frac{1}{\zeta(s+1)} \frac{(2\pi)^{s+1}}{\Gamma(s+1)}\left(\frac{k}{s} + \text{entire}\right) = k \Leftrightarrow (-1)^k = +1 \quad ✓$$

These manipulations are formal. But why just. They can be rigorously verified by careful contour deformation!

$\boxed{\emptyset.}$ Final thing to note:

⊛ $E_k(\tau) \to 0$ as $\tau \to \tau_* = s_* + i t_*$. (of course they have zeros!)

⊛ Thus $\dfrac{\partial_\tau E_k(\tau)}{E_k(\tau)} \to \infty$ as $\tau \to \tau_*$!

⊛ But $\langle T(\tau)\rangle = \sum_{n=1}^{\infty} \left(\sum_{m|n} m\, d_k(m) \right) q^n$ and $|q^n| = e^{-2\pi n t}$.

So to have a pole $\in \langle T(\tau)\rangle$, we must have $|d_k(m)| \sim e^{+2\pi m t_*}$.

$\downarrow$

Hagedorn growth!

⊛ By using $\int_\Gamma$-definition of $R.\sum_n d_k(n)\, n^{-s}$, we were able to extract finite parts of exponentially diverging infinite sums!

$\downarrow$

⊛ This is a whole story in and of itself. But, personally, I am very interested in developing meromorphic modular forms, which have poles of Hagedorn growth, as toy models for QCD models w/ Hagedorn growth.

⊛ Here, $E_{CAS} = \frac{1}{2} \sum_n E_n\, d(E)$ w/ $d(E) \sim e^{\beta_H E}$.

I do not know of another way to do this.

⊛ Here, the intuition from q-invariance ⇒ correct results, rather within math, that needed new structure c-ts for $f \in F_k$!

$\boxed{A.}$ ~~Related~~ So lattices & sum-rules are an important source of consistency for this story.

$\boxed{B.}$ As mentioned, $(-1)^{\sum_i \text{K.K. zero-modes}} = (-1)^k$ for modular forms comes from TWO independent sources:

$\boxed{C.}$ ⊛ L-functions for meromorphic modular forms:

- Let $Z(\tau)$ be a MF (holomorphic/weakly holomorphic) (or even meromorphic!) of weight $-k\in\frac{1}{2}\mathbb{Z}$, w/ q-series $Z(\tau) = q^{-\Delta}\left(1+\sum_{n\geq 1}^{\infty} c(n)\,q^n\right)$, $c(n)\in\mathbb{Z}\ \forall n\geq 1$.
- Then $Z(\tau) = q^{-\Delta}\prod_{n=1}^{\infty}(1-q^n)^{d(n)}$ w/ $d(n)\in\mathbb{Z}\ \forall n\geq 1$
- Then $\langle T(\tau)\rangle := \frac{1}{2\pi i}\frac{d}{d\tau}\log(Z(\tau))$ has an associated L-function and ζ-function $\zeta_{\langle T\rangle}(s) =_R \sum_{n=1}^{\infty}\frac{d(n)}{n^s} = \frac{(2\pi)^{s+1}}{\Gamma(s+1)}\frac{L^*_{\langle T\rangle}(s)}{\zeta(s+1)}$
- Where $\zeta_{\langle T\rangle}(s)$ has special values $\begin{cases}\zeta_{\langle T\rangle}(-1) = -2\Delta \\ \zeta_{\langle T\rangle}(0) = k\end{cases} \Rightarrow \begin{cases}-\Delta = E_{CAS}! \\ (-1)^k = e^{i\gamma_R}!\end{cases}$

1806.{αβγδ}3 + 1806.{αβγδ}4

$\boxed{D.}$ ⊛ Homomorphisms associated w/ $\widetilde{GL_2}(\mathbb{Z})$:

- Let $Z(\tau)$ be a MF (holomorphic/wk. holo./mero (even!)) of weight $k\in\frac{1}{2}\mathbb{Z}$ and $q^{-\Delta}\left(1+\sum_{n\geq 1}c(n)q^n\right)$ and w/ $\begin{cases}Z(\tau+1) = \rho\left(\widetilde{\begin{pmatrix}1&1\\0&1\end{pmatrix}}\right)Z(\tau)\\ Z\left(-\frac{1}{\tau}\right) = \rho\left(\widetilde{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}\right)\tau^k\, Z(+\tau)\end{cases}$ w/ $\rho: \widetilde{SL_2}(\mathbb{Z})\to\mathbb{C}^*$.
- Then $\exists$ ways to extend ρ to a homomorphism from $\widetilde{GL_2}(\mathbb{Z})$ to $\mathbb{C}^*$ (or $\mathbb{C}^*\oplus\mathbb{C}^*$), and w/ $\rho\left(\widetilde{\begin{pmatrix}-1&0\\0&1\end{pmatrix}}\right)^2 = \rho\left(\widetilde{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}\right)^4 = (-1)^{2k}$

1806.{αβγδ}3 + 1806.{αβγδ}5

$\widetilde{\mathrm{IV}}$. Brief Sketch of Conformalism in $CFT_2 / QFT_{d>2}$

$\boxed{\widetilde{\mathrm{IV}}.2}$ $\boxed{15(14)}$ M:✓ C:X

D. (continued...) $\boxed{\boxed{1806.\{\alpha\beta\gamma\delta\}3} + \boxed{1806.\{\alpha\beta\gamma\}5}}$

- Thus $\rho\left(\widetilde{\begin{pmatrix}-1&0\\0&1\end{pmatrix}}\right) = (-1)^k$, which follows from general facts about the metaplectic pre-image of $S = \begin{pmatrix}0&-1\\1&0\end{pmatrix} \in SL_2(\mathbb{Z})$, which sends $\tau \to -\frac{1}{\tau}$.

E. ~~Again, at the~~ This was purely mathematical. That was the point.

But now, back to Physics!

F. First: CFT_2 on $\mathbb{C}/\Lambda(\tau)$! Again, the path integral cannot depend on partition on $T^2_\tau = \mathbb{C}/\Lambda(\tau)$ w/ $\Lambda(\tau) := \{m+n\tau \mid m,n \in \mathbb{Z}\}$. Thus, $Z(\tau)$ is (or should be!) an explicit function of the lattice, $\Lambda(\tau)$. Thus $Z(\tau) = Z(\Lambda(\tau))$, and if $\Lambda(\tau) = \omega(\tau)\Lambda(\tau')$, then $Z(\Lambda(\tau)) = Z(\Lambda(\tau'))$!

BUT this fails! Sometimes $Z(\Lambda(\tau')) = e^{i\gamma(\tau|\tau')} Z(\Lambda(\tau))$.

⊛ Example: free (left moving) scalar CFT on T^2: $Z(\tau) = \frac{1}{q^{1/24}} \prod_{n\geq 1} \frac{1}{1-q^n}$ has $Z(\tau+1) = e^{-i\pi/12} Z(\tau)$ and $Z(-\frac{1}{\tau}) = \frac{1}{\sqrt{-i\tau}} Z(\tau) = \frac{1}{e^{-i\pi/4}} \frac{Z(\tau)}{\tau^{1/2}}$ $\longleftrightarrow$ $\boxed{\rho\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right) = \sqrt{\rho\left(\begin{pmatrix}-1&0\\0&1\end{pmatrix}\right)}}$

⊛ We expect to call the $\{e^{i\gamma_S}, e^{i\gamma_T}, e^{i\gamma_R}\}$ non-invariance phases Global Gravitational Anomalies.

⊛ $e^{i\gamma_R} = (-1)^{\sum KK\ 0\text{-modes}}$ $\Longleftrightarrow$ variation in $D\phi$ measure $\rightsquigarrow$ GGA! (I)

$\Longleftrightarrow$ S & T phases, which ARE GGAs! (II)

& $S \sim T \sim R$ fns w.r.t. $\Lambda(\tau) \to \omega\Lambda(\tau)$! (III)

G. But we might want to understand $\beta \to -\beta$ continuously. ~~Namely~~

⊛ Naively, $\exists$ problem: $\beta \in \mathbb{R} \Rightarrow \beta \to -\beta$ MUST contain the point

$\beta = 0$!!, which is BAD. $\beta = 0 \Leftrightarrow T = \infty$, and HERE BE DRAGONS!

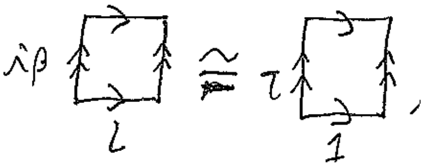
⊛ "standard" "trick" / cheat: $\beta \in \mathbb{R} \hookrightarrow \mathbb{C} \simeq \mathbb{R}^2$. Now, $\beta \to e^{i\theta}\beta \to -\beta$

while keeping $|\beta e^{i\theta}| = \beta$ unchanged. We avoid the singular configuration of the thermal identification BY ANALYTIC CONT.!

⊛ But does this work in $QFT_{d>1}$ / CFT_2?

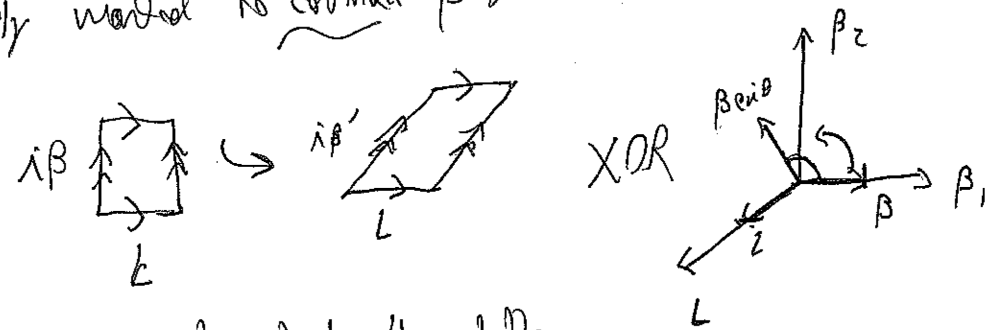
H. My claim: YES – or at least it's plausible.

⊛ Rather than write $\tau = \frac{i\beta}{L}$ for



keep β and L separate. Now, $(\beta, L) \in \mathbb{R}^2$, of course. Note that if we really wanted to continue β, there are "two" ways to do it!

⊛



I propose to study the latter.

⊛ At present, the picture is far from fully fleshed-out.

$\boxed{\text{I}}$ But $\exists$ a very encouraging & deeply fascinating picture emerging.

(*) SHO = $QFT_{d=1}$: $\beta \to e^{i\theta}\beta \to -\beta$. In order to preserve this,

at the level of modes we use a Bogoliubov transf.!

- $\begin{cases} a^\dagger \\ a \end{cases} \longrightarrow \begin{cases} a^\dagger(\theta) = \cos(\theta/2)\, a^\dagger - i\sin(\theta/2)\, a \\ a(\theta) = -i\sin(\theta/2)\, a^\dagger + \cos(\theta/2)\, a \end{cases} \Rightarrow [a, a^\dagger] = [a(\theta), a^\dagger(\theta)] = 1 \quad \forall\, \theta$

- $H(\theta) := \frac{1}{2}\{a(\theta), a^\dagger(\theta)\}$ has spectrum unchanged

- $H(\theta)(\beta e^{i\theta}) = \frac{1}{2}(e^{-i\theta}x^2 + e^{+i\theta}p^2)\,\beta\, e^{+i\theta} = \frac{1}{2}(x^2 + e^{2i\theta}p^2)\beta$

$$H\beta \to H(\theta)\, e^{i\theta}\beta \to H(\pi)(-\beta) = +\beta H \ @\ \theta = \pi!$$

- Again, a classical symmetry of $S_E[\phi(t)]_\beta$, albeit a silly looking one!

(*) Free scalar CFT_2: $L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{-m}\, a_{n+m} \to L_n(\theta) = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{-m}(\theta)\, a_{n+m}(\theta)$

- Note:
  - $L_n \to L_n(\theta) \to L_n(\pi) = -L_{-n}$ (
  - $[L_n, L_m] = (n-m)\, L_{n+m} + c\text{-term}$

    $\downarrow$
  - $[-L_{-n}, -L_{-m}] = (n-m)(-L_{-(n+m)}) = (-n-(-m))\, L_{-(n+m)}$
  - A kind of outer automorphism that preserves character.

Sections $\widetilde{\text{V}}$ and $\widetilde{\text{VI}}$ of T-Ref I and T-Ref II, respectively + unpublished. $\longleftrightarrow$

- Thus, if we continue naively in this free field representation, it really looks like continuation in this healthier limit/way can work! Reminds mapping tori?

IV. 2d χCFTs and Lattices $\boxed{\text{IV.1}}$

$\boxed{1\$(13)}$ M:X CV

$\boxed{A.}$ Recall the left-moving massless free scalar CFT on $S^1_L \times S^1_\beta$. It's g. PF for the full Fock space is
$$Z_{GC}(\beta) = q^{E_{VAC}} \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

$\boxed{B.}$ On general grounds, we expect $Z(\beta)$, the path integral, to depend only on the lattice of identified points $\Lambda(\beta, L) = \{im\beta + nL \mid m,n \in \mathbb{Z}\}$ where $z \sim z + im\beta + nL$ is on the complex plane. I.e. $Z = Z(\Lambda)$.

$\boxed{C.}$ Because this is a CFT, if two lattices Λ and Λ' are proportional, $\Lambda = \omega\Lambda'$ for some $\omega \neq 0$ ($\omega \in \mathbb{C}$), then we expect $Z_\Lambda = Z_{\Lambda'}$. (Scale invariance.)

~~$\boxed{D.}$ Now take At~~

$\boxed{D.}$ Now, rather than looking @ $S^1_L \times S^1_\beta$, look @ $T^2_\tau := \mathbb{C}/\Lambda(\tau)$, w/ $\Lambda(\tau) := \{m + n\tau \mid m,n \in \mathbb{Z}, \operatorname{Im}(\tau) \neq 1\}$. Clearly, when $i\beta/L = \tau$, $\Lambda(\beta, L) = \Lambda(\tau)$.

⊛ Note: $\Lambda(\tau) = \Lambda(\tau+1) = \Lambda(-1/\tau)\cdot\tau$. Demanding
$$Z(\tau) = Z(-1/\tau) = Z(\tau+1) \Leftrightarrow \text{demanding "modular invariance"}$$

⊛ But there is more! $\Lambda(\tau) = \Lambda(-\tau)$. Demanding this $\Leftrightarrow Z(\tau) = Z(-\tau)$
$\Leftrightarrow$ demanding τ-reflection invariance!

$\boxed{\text{At Chicago}}$ ✗ McGill ✓ X

IV. 2d χCFTs and Lattices

E. At this level, $\tau \underset{S}{\Leftrightarrow} -\frac{1}{\tau} \underset{T}{\Leftrightarrow} \tau+1 \underset{R}{\Leftrightarrow} -\tau$ are all REDUNDANT ways to encode the compact two-torus geometry into the path integral.

F. If the path integral depends on this redundant choice, then ~~that~~ we have a system that explicitly depends on an unphysical choice: a kind of anomaly under "large diffs"!

⊛ This applies, and has led to important consistency results, for S & T in
- "Heterotic ST 2" – Gross, Harvey, Martinec + Rohm (1986)
- "Int. top. phase + mod. inv." – Ryu + Zhang (2012)

⊛ I would like to see whether it is consistent to do so for $R: \tau \to -\tau$.

G. Be more concrete! $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$

⊛ $\eta(\tau) = e^{-i\pi/12}\, \eta(\tau+1)$ – EASY

⊛ $\eta(\tau) = \eta(-1/\tau)/\sqrt{-i\tau}$ – NOT-SO-EASY BUT OK

⊛ $Z(\tau) = \dfrac{1}{\eta(\tau)}$ = Path integral for left-moving scalar CFT on T^2.

⊛ Note: $E_{vac} = -\dfrac{1}{24} \Rightarrow Z_{GS}(\tau) = Z(\tau)$.

U Chicago ✓
McGill U X

$\boxed{\text{H.}}$ Focusing on T-phase, $\boxed{Z(\tau+1) = e^{i\gamma_T} Z(\tau) = e^{-i\pi/12} Z(\tau)}$.

We can only cancel this if we have 24 free (non trivial) scalars on T^2_τ. ~~Also~~ $\oint T = 26$ dimensions! $\boxed{\text{Hep-II and SPT did more thorough analysis.}}$

⊛ Fact: $$\left(\frac{1}{\eta(\tau)}\right)^{24} = \frac{1728}{E_4(\tau)^3 - E_6(\tau)^2} = \frac{1}{\Delta(\tau)} = \frac{1}{\Delta(\tau+1)} = \frac{1}{\tau^{12}}\frac{1}{\Delta(-1/\tau)}$$

⊛ Fact: $\frac{1}{\Delta(-\tau)} = \frac{1}{\Delta(\tau)}$, or $E_k(\tau) = \sum_{(m,n)\in\mathbb{Z}} \frac{1}{(m+n\tau)^k}$!

~~⊛ Real conditions~~

⊛ Fact: $$\begin{cases} \frac{1}{2} R. \sum_{n=1}^{\infty} n^1 \cdot 1 = -\frac{1}{24} = \frac{1}{2}\zeta(-1) \\ R. \sum_{n=1}^{\infty} n^0 \cdot 1 = -\frac{1}{2} = \zeta(0) \end{cases} \Rightarrow \frac{1}{\eta(\tau)} \to \frac{(-1)^{-1/2}}{\eta(\tau)} \text{ or } \tau \to -\tau$$

~~$\boxed{\text{I.}}$ At this stage, we $e^{i\gamma_R} = (-1)^{-1/2} =$~~

⊛ Fact: this modular form, $\frac{1}{\eta(\tau)}$, has weight $k = -\frac{1}{2}$:

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) (c\tau+d)^{1/2} \eta(\tau), \text{ where}$$

$\rho : \Gamma \to \mathbb{C}^\times$ is a homomorphism.

⊛ Fact: the sum-rules in the previous section

$\Rightarrow e^{i\gamma_R} = (-1)^{\zeta(0)} = (-1)^k$ for this Path Integral/MF!
$= (-1)^{\sum k \text{ zero-modes}}$!

$\boxed{\text{U Chicago ✓ McGill U X}}$

IV. 2d χCFTs and Lattices: $\boxed{\text{IV.4}}$

$\boxed{\text{I.}}$ The point: ⊛ Physics: $e^{i\gamma_S}, e^{i\gamma_T}, e^{i\gamma_R}$ all on same footing as explicit dependencies on unphysical choices: Global Gravitational Anomalies.

⊛ Formally: $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \Rightarrow f(-\tau) = (-1)^k f(\tau)$.

∃ tons of caveats, but...

A very important consistency test.

⊛ Mathematically: these phases are homomorphisms. By mandating that MFs $\in$ UHP extend as $E_k(\tau)$ do to DHP, we can use group theory alone to show ∃ two "unique" choices for $e^{i\gamma_R}$: $(\pm i)^{2k}$ or $(\pm i)^{4k}$.

⊛ Open question: Whether we MUST consider $\tau \to -\tau$ $(\beta \to -\beta)$, or whether we merely CAN insist on $\tau \to -\tau$ $(\beta \to -\beta)$?

⊛ If we MUST consider $\beta \to -\beta$, this $e^{i\gamma_A} = (-1)^{\xi k} = (-1)^k$ is a new GGA for generic finite-T QFTs.

⊛ Could lead to new classification!

... But it's tiny to rule-out NHO !!!

V. ... and everything else! $\boxed{\text{V.1}}$

$\boxed{\text{A}}$ Continuation vs. discrete reflections? $\boxed{\text{1711.abcde \& 1806.}\alpha\beta\gamma\delta\text{3...}}$ $\boxed{\text{Open}}$

$\boxed{\text{B}}$ Continuation ⇒ outer automorphisms of Virasoro! $\boxed{\text{1711.abcde...}}$

⇒ also of super-Virasoro? $\boxed{\text{Open}}$

$\boxed{\text{C.}}$ Perturbation theory? Is ~~F~~ reflection consistent, here? $\boxed{\text{1711.abcde...}}$ $\boxed{\text{Open}}$

$\boxed{\text{D.}}$ More rigorous derivation of Σ-rule and of $(\pm i)^{2k}$ – NO TIME

$\boxed{1806.\alpha\beta\gamma\delta 4 \text{ and } 1806.\alpha\beta\gamma\delta 5}$ (Higher genus = open!)

$\boxed{\text{E.}}$ Vacuum/Cosmic energies? ⊛ Already huge help in Large-N QCD!

⊛ Beyond that?

⊛ Tension w/ SUSY PFs?

⊛ CC problem???

$\boxed{\text{F.}}$ USING GGAs? Trust $e^{i\gamma_k}$ phases?

$\boxed{\text{G.}}$ Ubiquitous of resurgence outside Path Integrals (for MFs)?

$\boxed{\text{THANK YOU!}}$